

The number of seven digit integers..



M.N. Rao
Senior faculty,
Sri Chaitanya Educational
institutions

MODEL QUESTIONS

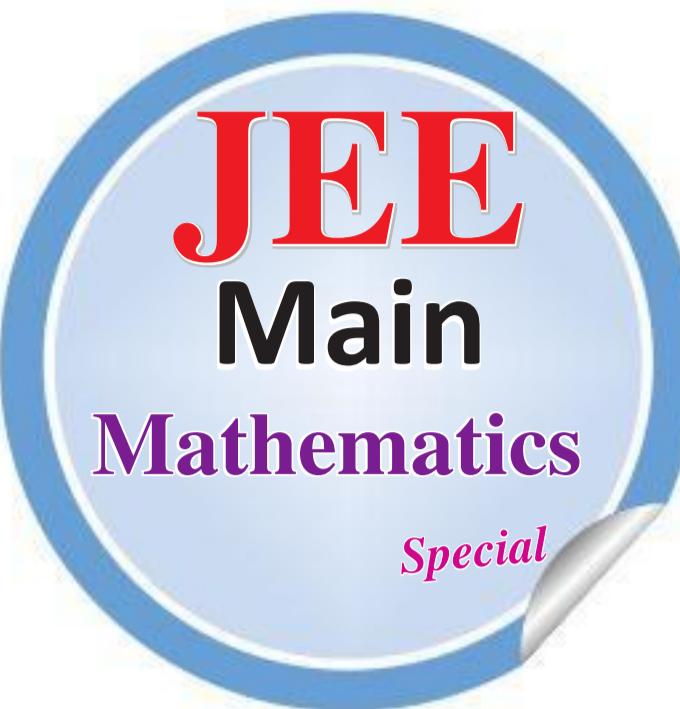
- $\frac{d}{dx} \left[\tan^{-1} \left(\frac{(3-x)\sqrt{x}}{1-3x} \right) \right] =$
 - $\frac{1}{2(1+x)\sqrt{x}}$
 - $\frac{3}{(1+x)\sqrt{x}}$
 - $\frac{2}{(1+x)\sqrt{x}}$
 - $\frac{3}{2(1+x)\sqrt{x}}$
- Suppose A_1, A_2, \dots, A_{30} are thirty sets each having 5 elements and B_1, B_2, \dots, B_n are n sets each with 3 elements. Let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^n B_j = S$ and each element of S belongs to exactly 10 of the A_i 's and exactly 9 of the B_j 's. Then n is equal to
 - 15
 - 3
 - 45
 - 90
- If $|z_1| = 2, |z_2| = 3, |z_3| = 4$ and $|z_1 + z_2 + z_3| = 5$ then $|4z_2 z_3 + 9z_2 z_3 + 16z_1 z_2| =$
 - 20
 - 24

KEY & HINTS

- 4; $\frac{d}{dx} [3 \tan^{-1} \sqrt{x}] = \frac{3}{(1+x)2\sqrt{x}}$
- 3; $S = \bigcup_{i=1}^{30} A_i$ so $\sin(s) = \frac{1}{10}(5 \cdot 30) = 15$
(since element in the union S belongs to exactly 10 of the A_i 's)
Again $S = \bigcup_{i=1}^n B_i$ so
 $n(s) = \frac{1}{9}(3 \cdot n) = \frac{n}{3} = 15 \Rightarrow n = 45$
- 4; $|4z_2 z_3 + 9z_2 z_3 + 16z_1 z_2| =$
 $= |z_1 \bar{z}_1 z_2 z_3 + \bar{z}_2 z_2 z_3 z_1 + \bar{z}_3 z_3 z_1 z_2|$
 $(4 = |z_1|^2 = z_1 \bar{z}_1)$
 $|z_1||z_2||z_3||z_1 + z_2 + z_3| = 120$

- 2; $T_r = \tan^{-1} \left(\frac{2^{r-1}}{1+2^{2r-1}} \right)$
 $= \tan^{-1} \left(\frac{2^r - 2^{r-1}}{1+2^r 2^{r-1}} \right)$
 $= t = \tan^{-1} (2^r) - \tan^{-1} (2^{r-1})$
 $S_n = \sum_{r=1}^n \tan^{-1} (2^r) - \sum_{r=1}^n \tan^{-1} (2^{r-1})$
 $= (\tan^{-1} 2 - \tan^{-1} 1) + (\tan^{-1} 4 + \tan^{-1} 2)$
 $+ \dots + (\tan^{-1} 2^n - \tan^{-1} 2^{n-1})$
 $= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$
- 3; As the roots are real and distinct $(a+1)^2 - 4(a^2 + a - 8) > 0$

- 48
- 120
- The sum of the infinite terms of the series $\tan^{-1} \left(\frac{1}{3} \right) + \tan^{-1} \left(\frac{2}{9} \right) + \tan^{-1} \left(\frac{4}{27} \right) + \dots$ is equal
 - $\frac{\pi}{6}$
 - $\frac{\pi}{4}$
 - $\frac{\pi}{3}$
 - $\frac{\pi}{2}$
- The value of a for which one root of the equation $x^2 - (a+1)x + a^2 + a - 8 = 0$ exceeds 2 and other is less than 2, are given by
 - $3 < a < 10$
 - $a \geq 10$
 - $-2 < a < 3$
 - $a \leq -2$
- Let $\Delta x = \begin{vmatrix} \cos^2 x & \cos x \sin x & -\sin x \\ \cos x \sin x & \sin^2 x & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$ then $\int_0^{\pi/2} [\Delta(x) + \Delta'(x)] dx$ equals
 - $\frac{\pi}{3}$
 - $\frac{\pi}{2}$
 - 2π
 - $\frac{3\pi}{2}$
- If the tangent at a point P , with parameter t , on the curve $x = 4t^2 + 3, y = 8t^3 - 1, t \in \mathbb{R}$, meets the



- curve again at a point Q , then the coordinates of Q are:
- $(t^2 + 3, -t^3 - 1)$
 - $(t^2 + 3, t^3 - 1)$
 - $(16t^2 + 3, -64t^3 - 1)$
 - $(4t^2 + 3, -8t^3 - 1)$
- Let P and Q be 3×3 matrices such that $P \neq Q$. If $P^3 = Q^3$ and $P^2Q = Q^2P$ then $\det(P^2 + Q^2)$ is equal to
 - 1
 - 0
 - 1
 - 2
 - The number of seven digit integers, with sum of the digits equal to 10 and formed by using the digits 1, 2 and 3 only, is
 - 55
 - 66

- 77
- 88
- $S = \{1, 2, 3, \dots, 20\}$ if 3 numbers are chosen at random from S , the probability for they are in A.P. is
 - $\frac{3}{38}$
 - $\frac{35}{33}$
 - $\frac{33}{35}$
 - $\frac{1}{38}$
- Suppose a, b, c are in A.P. and a^2, b^2, c^2 are in G.P. If $a < b < c$ and $a+b+c = 3/2$ then the value of a
 - $\frac{1}{2\sqrt{2}}$
 - $\frac{1}{2\sqrt{3}}$
 - $\frac{1}{2} - \frac{1}{\sqrt{3}}$
 - $\frac{1}{2} - \frac{1}{\sqrt{2}}$
- Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be positive increasing function with $\lim_{x \rightarrow \infty} \frac{f(3x)}{f(x)} = 1$. Then $\lim_{x \rightarrow \infty} \frac{f(2x)}{f(x)}$
 - $3/2$
 - 3
 - 1
 - $2/3$
- Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval $[-1, 1]$
 - $P(-1)$ is the minimum but $P(1)$ is not the maximum of P

$$\begin{aligned} & \int \frac{1+t^2}{t^{11/3}} dt \quad (t = \tan x) \\ & \int (t^{-11/3} + t^{-5/3}) dt \\ & = -\frac{3}{8} t^{-8/3} - \frac{3}{2} t^{-2/3} + C \\ & = -\frac{3}{8} \frac{(1+4\tan^2 x)}{\tan^2 x + \sqrt[3]{\tan^2 x}} + C \\ & \text{Thus } g(x) = -\frac{3}{8} \frac{(1+4\tan^2 x)}{\tan^2 x + \sqrt[3]{\tan^2 x}} \cdot g\left(\frac{\pi}{4}\right) \\ & = -\frac{15}{8} \end{aligned}$$

Clearly g is not defined at $x = 0$

$$\begin{aligned} 15. 2; I &= \int_0^\pi [\cot x] dx = \int_0^{\pi/2} [\cot x] + [\cot(\pi - x)] dx \\ &= \int_0^{\pi/2} [\cot x] + [-\cot x] dx \end{aligned}$$

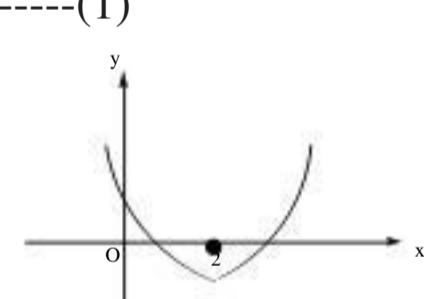
Put $\cot x = t$, so that

$$\begin{aligned} I &= \int_0^\infty [t] + [-t] \frac{dt}{1+t^2} \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \int [t] + [-t] \frac{dt}{1+t^2} \end{aligned}$$

But $[t] + [-t] = -1$ for $k-1 < t < k$, therefore

$$\begin{aligned} & \int_{k-1}^k ([t] + [-t]) \frac{dt}{1+t^2} \\ &= \int_{k-1}^k (-1) \frac{dt}{1+t^2} \\ &= [\tan^{-1} k - \tan^{-1}(k-1)] \\ & I = -\lim_{n \rightarrow \infty} \sum_{k=1}^n (\tan^{-1} k - \tan^{-1}(k-1)) \\ &= -\lim_{n \rightarrow \infty} [\tan^{-1} n - \tan^{-1} 0] = -\frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} 3a^2 + 2a - 33 &< 0 \\ (3a+11)(a-3) &< 0 \Rightarrow -\frac{11}{3} < a < 3 \end{aligned}$$



Also, for

$$2^2 - 2(a+1) + a^2 + a - 8 < 0$$

$$a^2 - a - 6 < 0 \Rightarrow (a-3)(a+2) < 0$$

$$-2 < a < 3 \quad (1)$$

From (1), (2), we get $-2 < a < 3$

6; Applying,

$$C_1 \rightarrow C_1 - \sin x \cdot C_3 \text{ and}$$

$$C_2 \rightarrow C_2 + \cos x \cdot C_3$$

we get

$$\Delta(x) = \begin{vmatrix} 1 & 0 & -\sin x \\ 0 & 1 & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$$

Applying

$$R_1 \rightarrow R_3 - \sin x R_1 + \cos x R_2, \text{ we get}$$

$$\Delta(x) = \begin{vmatrix} 1 & 0 & -\sin x \\ 0 & 1 & \cos x \\ 0 & 0 & \cos^2 x + \sin^2 x \end{vmatrix} = 1$$

$$\Rightarrow \Delta'(x) = 0$$

Thus,

$$\int_0^{\pi/2} [\Delta(x) + \Delta'(x)] dx = \int_0^{\pi/2} dx = \frac{\pi}{2}$$

$$7. 1; P(x = 4t^2 + 3, y = 8t^3 - 1)$$

$$\begin{aligned} & \text{Let } Q(4^2 + 3, 8y^3 - 1) \\ & \text{at } P_1 \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{24t^2}{8t} = 3t \\ & \text{tangent at } P \text{ is} \\ & y - 8t^3 + 1 = 3t(x - 4t^2 - 3) \end{aligned}$$

Q will satisfy it

$$t_1 = -\frac{t}{2}$$

$$Q(t^2 + 3, -t^3 - 1)$$

8; Take PQ common then det

9; Let x_i ($1 \leq i \leq 7$) be the digit at the i^{th} place. As $1 \leq x_i \leq 3$ and $x_1 + x_2 + \dots + x_7 = 10$. At most one x_i can be 3

Two cases arise:

Case 1:

Exactly one of x_i 's is 3. In this case exactly one of the remaining x_i 's is 2.

In this case, the number of seven digit numbers is

$$\frac{7!}{5!} = 7 \cdot 6 = 42$$

Case 2: None of x_i 's is 3

In this case exactly three x_i 's is 2 and the remaining four x_i 's are 1. In this case, the number of seven digit number is

$$\frac{7!}{3!4!} = 35$$

Hence, the required seven digit numbers is $42 + 35 = 77$

10. 1; required probability

$$\frac{3}{2(2n-1)} \text{ here } 2n = 20$$

11. 4;

We have

$$\begin{aligned} a+c &= 2b \Rightarrow 3b = a+b+c = \frac{3}{2} \\ \Rightarrow b &= \frac{1}{2} \end{aligned}$$

Thus, $a+c=1$ and $a^2c^2 = b^4 = \frac{1}{16}$

$$\Rightarrow 4a^2 - 4a + 1 = 0$$

$$\Rightarrow (2a-1)^2 = 0 \text{ or } (2a-1)^2 = 2$$

$$\Rightarrow a = \frac{1}{2} \text{ or } a = \frac{1}{2} \pm \frac{1}{\sqrt{2}}$$

As $a < b < c$ and $b = \frac{1}{2}$, we get

$$a = \frac{1}{2} - \frac{1}{\sqrt{2}}$$

12. 3; by definition

13. 4; $P(x) = 4x^3 + 3ax^2 = 2bx + c$

$$P'(0) = 0 \text{ So } c = 0$$

$$P'(x) = x(4x^2 + 3ax + 2b)$$

As $x = 0$ is the only real root of $P'(x) = 0$, roots of $4x^2 + 3ax + 2b$ must have imaginary roots,

Therefore $4x^2 + 3ax + 2b > 0 \forall x \in \mathbb{R}$

Thus $P'(x) < 0$ for $x < 0$

> 0 for $x > 0$

Therefore $x = 0$ is a point of local minimum at $x = 0$ as $P(-1) < P(1)$ we get $P(1)$ is maximum but $P(-1)$ is not minimum of P on $[-1, 1]$

14. 3; $f(x) = \lim_{n \rightarrow \infty} e^{x \tan(1/n) \log(1/n)}$